

PT: Lecture #16

①

Recall the definition of convolution of $f, g: \mathbb{R} \rightarrow \mathbb{R}$:

$$[f * g](x) = \int_{-\infty}^{\infty} f(y)g(x-y)dy$$

This is important for probability theory, because it allows us to consider sums of r.v.:

Theorem: (R.V. Sums & Convolution): Let \bar{X}, \bar{Y} be independent r.v. with density functions $f_{\bar{X}}(x)$ and $f_{\bar{Y}}(y)$, respectively. Then the r.v. $\bar{Z} = \bar{X} + \bar{Y}$ has density $f_{\bar{Z}}(z) = [f_{\bar{X}} * f_{\bar{Y}}](z)$.

Proof: Consider the cdf of \bar{Z} :

$$\begin{aligned} F_{\bar{Z}}(z) &= P(\bar{Z} \leq z) \\ &= P(\bar{X} + \bar{Y} \leq z) \end{aligned}$$

$$\text{Since } \bar{X}, \bar{Y} \text{ are independent} \rightarrow P(\bar{Y} \leq z - \bar{X})$$

$$= \int_{-\infty}^{\infty} f_{\bar{X}}(x) \cdot \left[\int_{-\infty}^{z-x} f_{\bar{Y}}(y) dy \right] dx$$

$$v = y - x$$

$$= \int_{-\infty}^{\infty} f_{\bar{X}}(x) \int_{-\infty}^z f_{\bar{Y}}(v-x) dv dx$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^z f_{\bar{X}}(x) f_{\bar{Y}}(v-x) dv dx$$

switch
order of
integration

$$= \int_{-\infty}^z \left[\int_{-\infty}^{\infty} f_{\bar{X}}(x) f_{\bar{Y}}(v-x) dx \right] dv$$

$$= \int_{-\infty}^z [f_{\bar{X}} * f_{\bar{Y}}](v) dv$$

$$\Rightarrow f_{\bar{Z}}(z) = f_{\bar{X}} * f_{\bar{Y}}(z)$$

So, to compute densities of sums of independent r.v., just convolve their densities!

ex: Let $\bar{X} \sim \mathcal{N}(\mu_X, \sigma_X^2)$
 $\bar{Y} \sim \mathcal{N}(\mu_Y, \sigma_Y^2)$

be independent normal. What can be said about their sum, $Z = \bar{X} + \bar{Y}$?

Well, $f_{\bar{X}}(x) = \frac{1}{\sqrt{2\pi\sigma_X^2}} \cdot \exp\left(-\left[\frac{(x-\mu_X)^2}{2\sigma_X^2}\right]\right)$

$$f_{\bar{Y}}(y) = \frac{1}{\sqrt{2\pi\sigma_Y^2}} \cdot \exp\left(-\left[\frac{(y-\mu_Y)^2}{2\sigma_Y^2}\right]\right)$$

S,

$$\begin{aligned}
 f_Z(z) &= \int_{-\infty}^{\infty} f_X(x) \cdot f_Y(z-x) dx \\
 &= \frac{1}{\sqrt{4\pi^2 \sigma_X^2 \sigma_Y^2}} \int_{-\infty}^{\infty} \exp\left(-\left[\frac{(x-\mu_X)^2}{2\sigma_X^2}\right]\right) \exp\left(-\left[\frac{(z-x-\mu_Y)^2}{2\sigma_Y^2}\right]\right) dx \\
 &= \frac{1}{2\pi \sqrt{\sigma_X^2 \sigma_Y^2}} \int_{-\infty}^{\infty} \exp\left(-\left[\frac{(x-\mu_X)^2}{2\sigma_X^2} + \frac{(z-x-\mu_Y)^2}{2\sigma_Y^2}\right]\right) dx \\
 &= \frac{1}{2\pi \sqrt{\sigma_X^2 \sigma_Y^2}} \int_{-\infty}^{\infty} \exp\left(-\left[\frac{\sigma_Y^2(x-\mu_X)^2 + \sigma_X^2(z-x-\mu_Y)^2}{2\sigma_X^2 \sigma_Y^2}\right]\right) dx \\
 &= \frac{1}{2\pi \sqrt{\sigma_X^2 \sigma_Y^2}} \int_{-\infty}^{\infty} \exp\left(-\left[\frac{\sigma_Y^2(x^2 - 2x\mu_X + \mu_X^2) + \sigma_X^2(z^2 + x^2 + \mu_Y^2 - 2zx - 2z\mu_Y + 2x\mu_Y)}{2\sigma_X^2 \sigma_Y^2}\right]\right) dx \\
 &= \frac{1}{2\pi \sqrt{\sigma_X^2 \sigma_Y^2}} \int_{-\infty}^{\infty} \exp\left(-\left[\frac{x^2[\sigma_Y^2 + \sigma_X^2] - 2x[\mu_X \sigma_Y^2 + \sigma_X^2(z - \mu_Y)] + \sigma_X^2[z^2 + \mu_Y^2 - 2z\mu_Y]}{2\sigma_X^2 \sigma_Y^2}\right]\right) dx
 \end{aligned}$$

Let $\sigma_Z = \sqrt{\sigma_X^2 + \sigma_Y^2}$. Then:

$$= \frac{1}{\sqrt{2\pi} \sigma_z^2} \cdot \frac{1}{\sqrt{2\pi} \frac{\sigma_x^2 \sigma_y^2}{\sigma_z^2}} \int_{-\infty}^{\infty} \exp \left[- \left(\frac{x - 2x \frac{\sigma_x^2 [z - \mu_y] + \sigma_y^2 [\mu_x]}{\sigma_z^2} + \frac{\sigma_x^2 [z + \mu_y - 2z \mu_y] + \sigma_y^2 [\mu_x]^2}{\sigma_z^2}}{2 \left(\frac{\sigma_x^2 \sigma_y^2}{\sigma_z^2} \right)} \right)^2 \right] dx$$

CTS

$$= \frac{1}{\sqrt{2\pi} \sigma_z^2} \cdot \frac{1}{\sqrt{2\pi} \frac{\sigma_x^2 \sigma_y^2}{\sigma_z^2}} \int_{-\infty}^{\infty} \exp \left[- \left(\frac{\left[x - \frac{\sigma_x^2 [z - \mu_y] + \sigma_y^2 [\mu_x]}{\sigma_z^2} \right]^2 - \left[\frac{\sigma_x^2 [z - \mu_y] + \sigma_y^2 [\mu_x]}{\sigma_z^2} \right]^2 + \frac{\sigma_x^2 [z + \mu_y - 2z \mu_y] + \sigma_y^2 [\mu_x]^2}{\sigma_z^2}}{2 \left(\frac{\sigma_x^2 \sigma_y^2}{\sigma_z^2} \right)} \right)^2 \right] dx$$

Inside the exponential, only the first factor depends on x . The rest are independent of x (depend only on z), and may be factored out.

$$S_{G_1} \textcircled{A} = \frac{1}{\sqrt{2\pi} \sigma_z^2} \exp \left(- \frac{\left[\frac{\sigma_x^2 [z - \mu_y] + \sigma_y^2 [\mu_x]}{\sigma_z^2} \right]^2 + \left[\frac{\sigma_x^2 [z + \mu_y - 2z \mu_y] + \sigma_y^2 [\mu_x]^2}{\sigma_z^2} \right]}{2 \left(\frac{\sigma_x^2 \sigma_y^2}{\sigma_z^2} \right)} \right) \textcircled{I}$$

$$\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi} \frac{\sigma_x^2 \sigma_y^2}{\sigma_z^2}} \exp \left(- \frac{\left[x - \frac{\sigma_x^2 [z - \mu_y] + \sigma_y^2 [\mu_x]}{\sigma_z^2} \right]^2}{2 \left(\frac{\sigma_x^2 \sigma_y^2}{\sigma_z^2} \right)} \right) dx \textcircled{II}$$

We conclude by noting that

$$\textcircled{I} = \frac{1}{\sqrt{2\pi} \sigma_z^2} \exp \left(- \left[\frac{(z - [\mu_x + \mu_y])^2}{2 \sigma_z^2} \right] \right)$$

$$\textcircled{II} = 1,$$

Hence, $f_Z(z) = \frac{1}{\sqrt{2\pi\sigma_z^2}} \exp\left(-\left[\frac{(z - [\mu_X + \mu_Y])^2}{\sigma_z^2}\right]\right)$, i.e.  (5)

~~$Z \sim$~~ $N(\mu_X + \mu_Y, \sigma_X^2 + \sigma_Y^2)$.

Something we saw is that for the Gaussian, adding ~~the two~~ together two independent Gaussians increased the variance: $\geq \max\{\sigma_X^2, \sigma_Y^2\}$.

But, we know if we average two Gaussians, then the variance decreases:

$$\begin{aligned} Z &= \frac{1}{2} [X + Y] \\ &= \frac{1}{2} X + \frac{1}{2} Y \\ &\quad \underbrace{\qquad\qquad\qquad}_{N(\frac{1}{2}\mu_X, \frac{1}{4}\sigma_X^2)} \quad \underbrace{\qquad\qquad\qquad}_{N(\frac{1}{2}\mu_Y, \frac{1}{4}\sigma_Y^2)} \\ &= N\left(\frac{1}{2}[\mu_X + \mu_Y], \frac{1}{4}[\sigma_X^2 + \sigma_Y^2]\right) \end{aligned}$$

So, if X, Y are ~~two~~ iid Gaussians (so, $\mu_X = \mu_Y$, $\sigma_X^2 = \sigma_Y^2$),

$Z \sim N(\mu, \frac{1}{2}\sigma^2)$... variance decreasing. Exploiting this is the basis of

modern statistics.

In general, we want to argue averages from random samples concentrate around their average value. This can be showed using concentration inequalities. (6)

The ur-example of a concentration inequality is Chebyshev's inequality:

Theorem (Chebyshev, discrete): Let \underline{X} be a discrete r.v. with finite variance and expectation.

Then $\forall \varepsilon > 0$, $P(|\underline{X} - E(\underline{X})| > \varepsilon) \leq \frac{\text{Var}(\underline{X})}{\varepsilon^2}$.

Proof: Let $m(x) = P(\underline{X}=x)$. Then notice

$$\text{Var}(\underline{X}) = \sum_x (x - E(\underline{X}))^2 m(x)$$

$$\geq \sum_{\substack{x \text{ s.t.} \\ |x - E(\underline{X})| \geq \varepsilon}} (x - E(\underline{X}))^2 m(x)$$

$$\geq \sum_{\substack{x \text{ s.t.} \\ |x - E(\underline{X})| \geq \varepsilon}} \varepsilon^2 \cdot m(x)$$

$$= \varepsilon^2 \sum_{\substack{x \text{ s.t.} \\ |x - E(\underline{X})| \geq \varepsilon}} m(x)$$

$$= \varepsilon^2 \sum_{\substack{x \text{ s.t.} \\ |x - E(\underline{X})| \geq \varepsilon}} m(x)$$

$$= \varepsilon^2 \cdot P(|X - E(X)| \geq \varepsilon)$$

$$\Rightarrow P(|X - E(X)| \geq \varepsilon) \leq \frac{\text{Var}(X)}{\varepsilon^2}$$