Lecture # 10: Optimal Transport

. With this Lemma in hand, we can now establish that J can be optimized using convex conjugates (φ, φ^*).

Theorem (Existence of Optimal Pair of Convex Conjugate Functions for J):
Let $\mu, \nu \in P(\mathbb{R}^d)$ with finite second moments. Let $\tilde{\Phi} = \{(\varphi, \psi) \in L^1(d\mu) \times L^1(d\nu) \mid x \cdot y \le \varphi(x) + \psi(y)$ μ-a.e. x, ν-a.e. $y\}$. Let $J[\varphi, \psi] = \int \varphi(x)\,d\mu(x) + \int \psi(y)\,d\nu(y)$.
Then $\exists\ (\varphi, \varphi^*)$ l.s.c., proper, conjugate convex functions $\mathbb{R}^d$ s.t. $\mathbb{R}^d$

$$\inf_{(\varphi,\psi) \in \tilde{\Phi}} J[\varphi, \psi] = J[\varphi, \varphi^*]. \qquad \in \tilde{\Phi}$$

Proof: Suppose first that μ, ν are compactly supported. Let $\{(\varphi_k, \psi_k)\}_{k=1}^{\infty}$ be a minimizing sequence for J on $\tilde{\Phi}$, i.e.

$$\lim_{k \to \infty} J[\varphi_k, \psi_k] = \inf_{(\varphi,\psi)} J[\varphi, \psi].$$

Lemma (Double Convexification) applies to the sequence $\{(\varphi_k, \psi_k)\}_{k=1}^{\infty}$, so let $\{(\bar{\varphi}_k, \bar{\psi}_k)\}_{k=1}^{\infty} = \{(\varphi_k^{**} - a_k, \varphi_k^* + a_k)\}_{k=1}^{\infty}$ be a minimizing sequence with the bounds guaranteed by the Lemma. Since

$$\bar{\psi}_k(y) = \sup_{x \in X} [x \cdot y - \varphi_k(x) - a_k],$$

$$\|\bar{\psi}_k(y_1) - \bar{\psi}_k(y_2)\| = \|\sup_{x \in X}\{x \cdot y_1 - \varphi_k(x) - a_k\} - \sup_{x \in X}\{x \cdot y_2 - \varphi(x) - a_k\}\|$$

$$\le \underbrace{\sup_{x \in X} \|x\|}_{< \infty} \cdot \|y_1 - y_2\|$$

$\Rightarrow \|\bar{\psi}_k\|_{Lip(Y)} \leq \sup_{x \in X} \|x\| \quad \forall k \Rightarrow \{\bar{\psi}_k\}_{k=1}^{\infty}$ are uniformly Lipschitz ②

Similarly, $\{\bar{\varphi}_k\}_{k=1}^{\infty}$ is uniformly Lipschitz.

We can in fact get more by noting that by the bounds of the double convexification lemma ensure that for k large enough, $\exists\, x_k \in X$, $y_k \in Y$ s.t.

$$-\sup_{x \in X} \overbrace{\frac{\|x\|^2}{2} \leq \bar{\varphi}_k(x_k)}^{\text{lower bound}} \leq \sup_{x \in X} \frac{\|x\|^2}{2} + \inf_{\tilde{\Phi}} J + M_2 + 1$$

$$-\sup_{y \in Y} \frac{\|y\|^2}{2} \leq \bar{\psi}_k(y_k) \leq \sup_{y \in Y} \frac{\|y\|^2}{2} + \inf_{\tilde{\Phi}} J + M_2 + 1$$

Now, X, Y compact $\Rightarrow$ the bounds are finite. They also hold independently of k, and so since the $\{\bar{\varphi}_k\}_{k=1}^{\infty}$, $\{\bar{\psi}_k\}_{k=1}^{\infty}$ are uniformly Lipschitz, we get that these sequences are in fact uniformly bounded.

By Arzelá-Ascoli (Royden & Fitzpatrick, Chapter 10), $\exists$ a subsequence $\{\bar{\varphi}_{\tilde{k}}\}_{k=1}^{\infty} \xrightarrow{C_b(X)} \bar{\varphi}$, same with $\bar{\psi}$ in $C_b(Y)$. Moreover, $J[\bar{\varphi}, \bar{\psi}] = \lim_{k\to\infty} J[\bar{\varphi}_k, \bar{\psi}_k]$, so $(\bar{\varphi}, \bar{\psi})$ is optimal. Finally, we can set $\bar{\varphi}$ off X to be $+\infty$ (same $\bar{\psi}$) and use double convexification to get the desired (φ, φ^*). ∎

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. The key point is to get an optimal $[\bar{\varphi}, \bar{\psi})$, then we can use double convexification.