OT: Lecture 18

$p \geq 1$ characterize

- We have established that for two measures $\mu, \nu \in \mathcal{P}_p(\mathbb{R}^d)$, we can ~~define~~ a constant-speed geodesic between μ and ν in $\mathcal{P}_p(\mathbb{R}^d)$ through McCann interpolation, namely letting $\pi^* = \underset{\pi \in \Pi(\mu,\nu)}{\arg\min} \int_{\mathbb{R}^d \times \mathbb{R}^d} \|x-y\|^p \, d\pi(x,y)$

$$P_t : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d$$

$$(x,y) \mapsto [1-t]x + ty$$

$$\mu_t := [P_t]_\# \pi^* \in \mathcal{P}_p(\mathbb{R}^d)$$

Then $t \mapsto \mu_t$ is an a.c. curve with $\mu_0 = \mu$, $\mu_1 = \nu$, $W_p(\mu_t, \mu_s) = |t-s| \cdot W_p(\mu, \nu)$.

- We can leverage this to develop a <u>dynamical</u> characterization of optimal transport. The idea is to realize Wasserstein distance as energy-minimizing paths through distributions & velocity fields that satisfy the continuity equation.

- To make this precise, we require a notion of "energy," or really a functional over measures. This is given by the Benamou-Brenier functional as follows:

Lemma: Let $p, q \geq 1$ be such that $\frac{1}{p} + \frac{1}{q} = 1$. Define $K_q := \{(a,b) \in \mathbb{R} \times \mathbb{R}^d \mid a + \frac{1}{q}|b|^q \leq 0\}$. For $(t,x) \in \mathbb{R} \times \mathbb{R}^d$, we have

$$f_p(t,x) := \sup_{(a,b) \in K_q} [at + b \cdot x] = \begin{cases} \frac{1}{p} \frac{|x|^p}{t^{p-1}}, & \text{if } t > 0 \\ 0, & t = 0, x = 0 \\ +\infty, & \text{else} \end{cases}$$

Proof: If $t>0$, then we want to take a as large as allowed by the constraint $(a,b)\in K_q$, namely $a = -\frac{1}{q}|b|^q$. Then

$$f_p(t,x) = \sup_{(a,b)\in K_q} [at + b\cdot x]$$

$$= \sup_b \left[-\frac{1}{q}|b|^q \cdot t + b\cdot x\right]$$

$$= t\cdot \sup_b \left[-\frac{1}{q}|b|^q + b\cdot\left(\frac{x}{t}\right)\right]$$

Since the LT of $b \mapsto \frac{1}{q}|b|^q$ is $y \mapsto \frac{1}{p}|y|^p$, we get

$$\sup_b \left[-\frac{1}{q}|b|^q + b\cdot y\right]$$

$$= \frac{1}{p}|y|^p$$

$$\Rightarrow t\cdot \sup_b \left[-\frac{1}{q}|b|^q + b\cdot\left(\frac{x}{t}\right)\right]$$

$$= t\cdot \frac{1}{p}\cdot\left|\frac{x}{t}\right|^p$$

$$= \frac{1}{p}\frac{|x|^p}{t^{p-1}},$$ as needed.

The cases $t\leq 0$ are easy. ∎

Corollary: f_p is convex and l.s.c.

Proof: f_p is the supremum of linear functions. ∎

· Now, we can define an appropriate energy functional $\mathcal{B}_p$.

Defn: Let ρ be a measure on some space X ~~and~~ $\mathcal{M}(X)$ and let E be an element of $\mathcal{M}^d(X)$. Then we define

$$\mathcal{B}_p(\rho, E) := \sup\left\{ \int_X a(x)\,d\rho(x) + \int_X b(x)\cdot dE(x) \,\middle|\, (a,b) \in C_b(X, K_q) \right.$$

Proposition: $\mathcal{B}_p$ is (i) convex
(ii) l.s.c. on $\mathcal{M}(X) \times \mathcal{M}^d(X)$
(iii) non-negative

Theorem (Benamou-Brenier Formulation of OT): Let $\Omega \subset \mathbb{R}^d$ be compact and convex. Let $\mu, \nu \in \mathcal{P}(\Omega)$. Let ρ, E be considered as measures on $\Omega \times [0,1]$. Then

$$W_p^p(\mu,\nu) = \min\left\{ \mathcal{B}_p(\rho, E) \,\middle|\, \partial_t \rho + \nabla \cdot E = 0,\ \rho_0 = \mu,\ \rho_1 = \nu \right\}$$

where ρ_0, ρ_1 are the restriction (naturally) to $t = 0, 1$.

· So, we can think of W_p as minimizing $\mathcal{B}_p$ over curves ; ~~[illegible]~~ velocity fields satisfying ~~[illegible]~~ the continuity equation subject to data-imposed boundary conditions