OT: Lecture #19

- We would like to understand how functionals on $\mathcal{W}_p$ work. What kinds of things are we talking about? Generally, a functional on $\mathcal{W}_p$ is a function $F : \mathcal{W}_p \to \mathbb{R}$. Some specific types of functional we might consider are:
  - potential energy: for some $V : \Omega \to \mathbb{R}$, $\mathcal{V}(\mu) := \int_\Omega V(x)\, d\mu(x)$.
  - interaction energy: for some $W : \Omega \times \Omega \to \mathbb{R}$, $\mathcal{W}(\mu) := \int_{\Omega \times \Omega} W(x,y)\, d\mu(x)\, d\mu(y)$
  - distance from a fixed point (measure): $\mu \mapsto W_p^p(\mu, \nu)$ for some fixed ν.
  - integral functional of the density: for some $f : \mathbb{R} \to \mathbb{R}$, and a.c. measure μ with density ρ, let $\mathcal{F}(\mu) := \int_\Omega f(\rho(x))\, dx$

- For these and other functionals, the aim is to solve problems roughly of the form

$$\boxed{\underset{\mu \in \text{constraints on } \mathcal{W}_p}{\arg\min} \; F(\mu)}$$

For this to be tractable, we want to understand issues of continuity & convexity first and foremost.

- Let us first establish continuity.

<u>Proposition (Continuity of Potential Energies)</u>: Let $V \in C_b(\Omega)$ and consider the functional $\mathcal{V}(\mu) := \int_\Omega V(x)\, d\mu(x)$. Then:

(i) $\mathcal{V} : \mathcal{P}(\Omega) \to \mathbb{R}$ is continuous w.r.t. weak convergence of probability measures.

(ii) If V is lower semi-continuous (l.s.c., meaning for every $x_n \to x$, $\liminf_{n\to\infty} f(x_n) \geq f(x)$), and bounded from below

Then $\mathcal{V}$ is ~~[illegible]~~ l.s.c. as well.

(iii) If V is not continuous (or semicontinuous), then $\mathcal{V}$ is not continuous (or semicontinuous).

Proof: (i) This is immediate from the weak convergence of measures definition. Indeed, if $\{\mu_n\}_{n=1}^{\infty} \to \mu$ weakly, then since $V \in C_b(\Omega)$ is an allowed test function, we have

$$\lim_{n\to\infty} \underbrace{\int_\Omega V(x)\,d\mu_n(x)}_{:= \mathcal{V}(\mu_n)} = \underbrace{\int_\Omega V(x)\,d\mu(x)}_{:= \mathcal{V}(\mu)}$$

as needed.

(ii) Suppose V is lsc and bounded from below. Then there exists a sequence of Lipschitz and bounded functions $\{V_k\}_{k=1}^{\infty}$ converging increasingly to V. Then by monotone convergence, we have

$$\mathcal{V}(\mu) = \int_\Omega V(x)\,d\mu(x)$$

$$= \lim_{k\to\infty} \int_\Omega V_k(x)\,d\mu(x)$$

$$= \sup_k \int_\Omega V_k(x)\,d\mu(x),$$

so that $\mathcal{V} : \mathcal{P}(\Omega) \to \mathbb{R}$ is a supremum of continuous functionals, hence itself l.s.c.

(iii) Suppose $\mathcal{V}$ is continuous. Then let us consider a sequence of points in Ω, $\{x_k\}_{k=1}^{\infty}$ converging to some x. Let $\mu_k = \delta_{x_k}$, $\mu = \delta_x$ be Dirac masses at these points. Then by continuity of $\mathcal{V}$, we have $\lim_{k \to \infty} \mathcal{V}(\mu_k) = \mathcal{V}(\mu)$ (this follows because $\mu_k \to \mu$ in the sense of weak convergence of measures). But,

$$\mathcal{V}(\mu_k) = \int_\Omega V(x)\,d\mu_k(x) = \int_\Omega V(x)\,\delta_{x_k}(x) = V(x_k),$$

and similarly, $\mathcal{V}(\mu) = V(x)$. So, continuity of V follows. Same argument establishes V is l.s.c. if $\mathcal{V}$ is. ■

- We can establish the continuity of interaction energies similarly; we exclude the proof for brevity.

<u>Proposition (Continuity of Interaction Energies)</u>: Let $W \in C_b(\Omega \times \Omega)$. Then $\mathcal{W}(\mu) := \int_{\Omega\times\Omega} W(x,y)\,d\mu(x)\,d\mu(y)$ is continuous with respect to weak convergence of probability measures. If W is l.s.c. and bounded from below, $\mathcal{W}$ is l.s.c.

- We now consider functionals of the form $\mu \mapsto W_p^p(\mu,\nu)$ for a fixed measure ν, and their generalizations.

<u>Proposition (Continuity of Transportation Functionals)</u>: Let $\nu \in \mathcal{P}(\Omega)$ be fixed.

(i) For any $p < +\infty$, the functional $\mu \mapsto W_p(\mu, \nu)$ is continuous w.r.t. weak convergence of measures μ if Ω is compact.

(ii) If Ω is not compact, then $\mu \mapsto W_p(\mu,\nu)$ is l.s.c.

(iii) For any cost function $c: \Omega \times \Omega \to \mathbb{R}$ that is continuous, $\mathcal{T}_c : \mathcal{P}(\mathbb{R}^d) \to \mathbb{R}$ given by

$$\mathcal{T}_c(\mu) = \min_{\gamma \in \Pi(\mu,\nu)} \int_{\Omega \times \Omega} c(x,y)\, d\gamma(x,y)$$

is continuous if Ω is compact.

(iv) If Ω is not compact, then if c is l.s.c. and bounded from below, then $\mathcal{T}_c$ is l.s.c.

<u>Proof</u>: (i) $\ulcorner$ Obvious, because W_p metrizes weak convergence; similar argument for (iii) $\lrcorner$

(iii)

(ii), (iv) $\ulcorner$ Since W_p is a special case of $\mathcal{T}_c$, we focus on the latter. Let $\{\mu_k\}_{k=1}^{\infty}$ converge weakly to μ. Let $\{\gamma_k\}_{k=1}^{\infty}$ be a sequence of optimal plans for (μ_k, ν) for a l.s.c., bounded from below cost $c: \Omega \times \Omega \to \mathbb{R}$. Extract a subsequence $\{\gamma_{k_h}\}_{h=1}^{\infty}$ s.t. (a.) $\lim_{h \to \infty} \mathcal{T}_c(\mu_{k_h}, \nu) = \liminf_{k \to \infty} \mathcal{T}_c(\mu_k, \nu)$;

(b.) $\{\gamma_{k_h}\}$ converges weakly to γ

Note that we have that $\gamma \in \Pi(\mu, \nu)$, and so

$$\tilde{\mathcal{T}}_c(\mu,\nu) = \inf_{\pi \in \Pi(\mu,\nu)} \int_{\Omega\times\Omega} c(x,y)\,d\pi(x,y)$$

$$\leq \int_{\Omega\times\Omega} c(x,y)\,d\gamma(x,y)$$

$$\leq \liminf_{h\to\infty} \int_{\Omega\times\Omega} c(x,y)\,d\gamma_{k_h}(x,y)$$

$$= \lim_{h\to\infty} \tilde{\mathcal{T}}_c(\mu_{k_h},\nu)$$

$$= \liminf_{k\to\infty} \tilde{\mathcal{T}}_c(\mu_k,\nu). \quad \blacksquare$$

• One other class of functionals we may wish to consider are <u>dual norms</u>, defined as follows. Letting $\mathcal{X}$ be a Banach space of functions on Ω, we define the dual norm as

$$\|\mu\|_{\mathcal{X}'} := \sup_{\substack{\varphi\in\mathcal{X}\\ \|\varphi\|\leq 1}} \int_{\Omega} \varphi(x)\,d\mu(x)$$

$$= \sup_{\varphi\in\mathcal{X}\setminus\{0\}} \frac{\int \varphi(x)\,d\mu(x)}{\|\varphi\|}.$$

<u>Proposition (Continuity of Dual Norms)</u>: Let $\mathcal{X}$ be a Banach space of functions on Ω, such that $\mathcal{X}\cap C_b(\Omega)$ is dense in $\mathcal{X}$. Then

$$\mu \mapsto \|\mu\|_{X'} = \sup_{\substack{\ell \in X \\ \|\ell\| \leq 1}} \int \ell(x)\, d\mu(x)$$

is l.s.c. (note, we define the functional to be $+\infty$ if $\mu \notin X'$).

Proof: Once again, we aim to write the functional as a supremum of continuous functionals. This follows by writing

$$\|\mu\|_{X'} = \sup_{\substack{\ell \in X \\ \|\ell\| \leq 1}} \int \ell(x)\, d\mu(x)$$

$$= \sup_{\substack{\ell \in X \cap C_b(X) \\ \|\ell\| \leq 1}} \underbrace{\int \ell(x)\, d\mu(x)}$$

each of these is now continuous, thanks to the restriction to $C_b(X)$. ∎