Optimal Transport: Lecture #8

- Let's continue our review of convex analysis.
- Recall that in $L^p(\mathbb{R}^d)$, convolution is a way of combining functions that leads to the preservation of good properties of either. If $f \in L^p(\mathbb{R}^d)$, $g \in L^q(\mathbb{R}^d)$, then $f * g \in L^r(\mathbb{R}^d)$, for $\frac{1}{p} + \frac{1}{q} = \frac{1}{r} + 1$, is defined as

$$f * g(x) = \int_{\mathbb{R}^d} f(y) g(x-y)\, dy$$

In particular, if $f \in L^1(\mathbb{R}^d)$ and $g \in L^q(\mathbb{R}^d)$, $f * g \in L^q(\mathbb{R}^d)$.

- Crucially, $\widehat{f * g} = \hat{f} \cdot \hat{g}$ where $\hat{f}(\gamma) = \int_{\mathbb{R}^d} f(x) \exp[-2\pi i \gamma \cdot x]\, dx$.

Noting the classical correspondence between smoothness of f and decay of $\hat{f}$ (see Evans or Lemma 1.2.3 in K. Gröchenig's "Foundations of Time Frequency Analysis"), we see that if g is "really nice", then so is $f * g$, even if f is "bad". So, we can think of convolution as a <u>regularizing</u> operation.

<u>Defn</u>: Suppose φ, ψ are proper convex functions. Then the <u>inf convolution</u> of φ, ψ is $[\varphi \,\square\, \psi](z) := \inf_{x + x' = z} \{\varphi(x) + \psi(x')\}$.

<u>Exercise (Interaction of $\square$ and Legendre Transform)</u>: For φ, ψ proper convex functions,

$$[\varphi \,\square\, \psi]^* = \varphi^* + \psi^*.$$

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Proposition (Legendre Duality for LSC): Let $\ell : \mathbb{R}^d \to \mathbb{R} \cup \{+\infty\}$ be a proper function. Then TFAE:

(i) ℓ is convex and l.s.c.

(ii) $\ell = \psi^*$ for some proper ψ

(iii) $\ell = \ell^{**}$

Proof: (iii) $\Rightarrow$ (ii): $\psi = \ell^*$

(ii) $\Rightarrow$ (i): ψ^* is automatically convex and l.s.c.

(i) $\Rightarrow$ (iii): Suppose ℓ is convex and l.s.c. We aim to show $\ell = \ell^{**}$. Note that by definition, explicated in last week's discussion of double convexification, $\ell(x) \geq \sup_y \{x \cdot y - \ell^*(y)\} = \ell^{**}(x)$. Now, suppose

$x \in \mathrm{Int}(\mathrm{Dom}(\ell))$. Since $\partial \ell(x) \neq \emptyset$, $\exists\, y \in \partial \ell(x)$ and then we know for such a y, $\ell(x) + \ell^*(y) = x \cdot y$ $\downarrow$ take sup over y

$$\Rightarrow \ell(x) \leq \sup_y \{x \cdot y - \ell^*(y)\}$$

$$= \ell^{**}(x)$$

$\Rightarrow \ell = \ell^{**}$ on $\mathrm{Int}(\mathrm{Dom}(\ell))$.

In general, we can use inf convolution to extend to points beyond $\mathrm{Int}(\mathrm{Dom}(\ell))$.

Indeed, let $\psi_\varepsilon(x) := \frac{1}{2\varepsilon}\|x\|_2^2$

$$\ell_\varepsilon := \ell \,\square\, \psi_\varepsilon$$

Then $\ell(x) = \lim_{\varepsilon \to 0^+} \ell_\varepsilon(x)$ and $\mathrm{Dom}(\ell_\varepsilon) = \mathbb{R}^d$. Thus, our above argument gives $\ell_\varepsilon^{**} = \ell_\varepsilon$. Note now $\ell_\varepsilon \le \ell$

$$\Rightarrow \ell_\varepsilon^* \ge \ell^*$$

$$\Rightarrow \ell_\varepsilon^{**} \le \ell^{**}$$

Hence, $\ell^{**}(x) \ge \liminf_{\varepsilon \to 0} \ell_\varepsilon^{**}(x) = \liminf_{\varepsilon \to 0} \ell_\varepsilon(x) = \ell(x)$. Since we know $\ell^{**}(x) \le \ell(x)$ in general, we see $\ell^{**} = \ell$ in general, so that (i) $\Rightarrow$ (iii) holds. ∎

- Lastly, we establish a connection between convexity of ℓ and smoothness of ℓ^* (Think like Fourier!). If ℓ is strictly convex in a neighborhood of $x \in \mathbb{R}^d$, then ℓ^* is differentiable on $\partial\ell(x)$ and moreover, $\nabla \ell^*(y) = x$ $\forall$ $y \in \partial \ell(x)$.

- This can be used to show that if ℓ is differentiable and strictly convex, then

$$[\nabla \ell]^{-1} = \nabla \ell^* \quad \text{(exercise)}$$

If further smoothness holds (e.g. ℓ, ℓ^* are twice differentiable), then

$$x = \nabla \ell^*(\nabla \ell(x))$$

$$\Rightarrow 1 = \nabla^2 \ell^*(\nabla \ell(x)) \cdot \nabla^2 \ell(x)$$

$$= D^2 \ell^*(\nabla \ell(x)) \cdot D^2 \ell(x)$$

$$\Rightarrow D^2 \ell^*(\nabla \ell(x)) = [D^2 \ell(x)]^{-1} \Big\} \text{ matrix inverse}$$