Math 290-01, Fall 2022 Optimal Transport: Theory and Applications

Instructor: Prof. James M. Murphy
Email: jm.murphy@tufts.edu
Lecture: MW 6-7:15, JCC 502
Office Hours: Wednesday 3pm-4pm; Fridays 11am-12pm

Course websites:
(1) canvas.tufts.edu
(2) https://jmurphy.math.tufts.edu/Teaching/Fall2022/MATH290/

Textbooks:

L. Ambrosio, N. Gigli, and G. Savare. "Gradient Flows in Metric Spaces and in the Space of Probability Measures." Second edition. 2008.
G. Peyré and M. Cuturi. "Computational Optimal Transport with Applications to Data Sciences". 2020.
F. Santambrogio. "Optimal Transport for Applied Mathematicians." 2015.
C. Villani. "Topics in Optimal Transport." Second Edition. 2016.

Background References:

J.J. Benedetto and W. Czaja. "Integration and Modern Analysis." 2009.
L.C. Evans. "Partial Differential Equations." Second Edition. 2010.
H.L. Royden and P.M. Fitzpatrick. "Real Analysis." Fourth Edition. 2010.

Prerequisites: MATH 235 (required); MATH 165-166 or equivalent (required); functional analysis and PDE (helpful)

Expectations for Participation

- All students are *required* to attend all lectures and participate in class.
- All students are *required* to attend the professor's office at least twice during the semester.

Modality: The course will generally be in person. There will be a few exceptions due to instructor travel and holidays. Remote classes will be recorded for asynchronous viewing.

Topics: Optimal transport (OT) is an old subject that has seen a renaissance in recent decades. The basic idea—dating to Monge in the 1780s—is to transport one probability measure to another by "moving mass" in a cost-minimizing way. This simple-sounding problem is remarkably deep, and has close connections with nonlinear PDE, functional analysis, convex optimization, and statistics. This course will be guided by Villani's well-known (shorter) text and will focus on understanding the mathematics of OT, including: Kantorovich duality, existence and uniqueness of OT plans and maps, Brenier's Theorem, as well as Wasserstein space and its geometry. We will also touch on work in the past decade around computational OT and its applications to data science, including linear programming, entropic regularization, and Wasserstein barycenters. Students can expect mostly theorems and proofs, with some digressions to discuss algorithms and applications. The course will assume background in measure theory and some basic familiarity with probability theory. Given the tight connection between OT and nonlinear PDE, some background in the mathematical theory of PDE (e.g. weak solutions, regularity) will be helpful. Background in coding is not required, but may enhance enjoyment of certain parts of the course. There will be few formal assignments, but students will be expected to give two presentations on major ideas, papers, and algorithms associated to OT. The class will be lecture-based but with ample time for discussion.

Rough Lecture Schedule:

Lecture 1 (Sept. 7): Basic formulation of Monge and Kantorovich problem
Lecture 2 (Sept. 12): Introduction to Kantorovich duality
Lecture 3 (Sept. 14): Proof of Kantorovich duality
Lecture 4 (Sept. 19): Discrete OT as linear program
Lecture 5 (Sept. 21): Entropic regularization and Sinkhorn's algorithm
Lecture 6 (Sept. 26): Existence of minimizers for quadratic cost
Lecture 7 (Sept. 28): Background on convex analysis
Lecture 8 (Oct. 3): Background on convex analysis
Lecture 9 (Oct. 5): Knott-Smith optimality/ Brenier's theorem
Lecture 10 (Oct. 12): Knott-Smith optimality/ Brenier's theorem
Lecture 11 (Oct. 17): Knott-Smith optimality/ Brenier's theorem
Lecture 12 (Oct. 19): OT on the real line
Presentation 1, Round 1 (Oct. 24)
Presentation 1, Round 2 (Oct. 26)
Lecture 13 (Oct. 31): Wasserstein distances
Lecture 14 (Nov. 2): Convergence in Wasserstein space
Lecture 15 (Nov. 7): Curves & geodesics in Wasserstein space
Lecture 16 (Nov. 9): Curves & geodesics in Wasserstein space
Lecture 17 (Nov. 14): Curves & geodesics in Wasserstein space
Lecture 18 (Nov. 16): Wasserstein barycenters
Lecture 19 (Nov. 21): Functionals on Wasserstein space
Lecture 20 (Nov. 28): Convexity in Wasserstein space
Lecture 21 (Nov. 30): Displacement convexity
Presentation 2, Round 1 (Dec. 5)
Presentation 2, Round 2 (Dec. 7)
Presentation 2, Round 3 (Dec. 12)

The course is proof-based, and background in analysis (MATH 235) and probability (MATH 165) is required. Background in PDE and functional analysis will be helpful, and to a lesser extent so will background in basic coding (e.g. scripting in MATLAB/Python).

Grading: There will be two in-class presentations. The first will be 20 minutes in length and the second will be 30. These will be done in pairs. There will be a separate list of possible papers and topics to present on.

Lecture Participation and Attendance: 15%
Office Hours Participation: 15%

Presentation 1: 30%
Presentation 2: 40%

Quarantine Considerations: If you become sick or must quarantine: We really hope everybody will stay well, but if you do become sick or have to quarantine and this interferes with your class participation, please let us know and we will work out a makeup schedule for you.

Accommodations for Students with Disabilities: Tufts University values the diversity of our students, staff, and faculty; recognizing the important contribution each student makes to our unique community. Tufts is committed to providing equal access and support to all qualified students through the provision of reasonable accommodations so that each student may fully participate in the Tufts experience. If you have a disability that requires reasonable accommodations, please contact the StAAR Center (formerly Student Accessibility Services) at StaarCenter@tufts.edu or 617-627-4539 to make an appointment with an accessibility representative to determine appropriate accommodations. Please be aware that accommodations cannot be enacted retroactively, making timeliness a critical aspect for their provision.