FoSML: Lecture 12

Recall our set-up

- $\Phi(u)$ is a convex, non-decreasing function with $\mathbb{1}_{u \le 0} \le \Phi(-u)$ $\forall u \in \mathbb{R}$.
- The Φ-loss of $h : \mathcal{X} \to \mathbb{R}$ at $(x,y) \in \mathcal{X} \times \{-1,1\}$ is $\Phi(-yh(x))$, and the expected loss is

$$\mathcal{L}_\Phi(h) = \mathbb{E}_{(x,y)\sim \mathcal{D}}\left(\Phi(-yh(x))\right) = \mathbb{E}_{x\sim \mathcal{D}_X}\Big(\eta(x)\Phi(-h(x)) + (1-\eta(x))\Phi(h(x))\Big).$$

- For any $x \in \mathcal{X}$, let $u \mapsto L_\Phi(x,u)$ be the function $L_\Phi(x,u) = \eta(x)\Phi(-u) + (1-\eta(x))\Phi(u)$,

so that $\mathcal{L}_\Phi(h) = \mathbb{E}_{x\sim \mathcal{D}_X}\big(L_\Phi(x, h(x))\big)$.

- Define $h^*_\Phi(x) = \underset{u\in[-\infty,\infty]}{\arg\min}\ L_\Phi(x,u) = \underset{u\in[-\infty,+\infty]}{\arg\min}\ \eta(x)\Phi(-u) + (1-\eta(x))\Phi(u)$.

This is not unique, so just pick one. Let $\mathcal{L}^*_\Phi = \mathbb{E}_{(x,y)\sim\mathcal{D}}\left(\Phi(-yh^*_\Phi(x))\right)$.

Proposition: Let Φ be a convex and non-decreasing function that is differentiable at 0 and satisfying $\Phi'(0) > 0$. Then the minimizer of Φ defines the Bayes classifier, i.e. for any $x \in \mathcal{X}$,

$$h^*_\Phi(x) > 0 \iff h^*(x) > 0, \qquad h^*_\Phi(x) = 0 \iff h^*(x) = 0.$$

In particular, $\mathcal{L}^*_\Phi = R^*$.

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Proof: Fix $x \in \mathcal{X}$. If $\eta(x)=0$, then $h^*(x) = -\frac{1}{2}$ and $h^*_\Phi(x) = -\infty$, so that $\operatorname{sgn}(h^*(x)) = \operatorname{sgn}(h^*_\Phi(x))$. Similarly, if $\eta(x)=1$, then $h^*(x)=\frac{1}{2}$ and $h^*_\Phi(x)=+\infty$, so that $\operatorname{sgn}(h^*(x)) = \operatorname{sgn}(h^*_\Phi(x))$ in this case as well.

Let u^* be the minimizer defining $h^*_\Phi(x)$. Recall that u^* is a minimizer of $u \mapsto L_\Phi(x,u) \Leftrightarrow$ the subdifferential of this function at u^* contains 0. More precisely, taking the subdifferential at u^* yields

$$\partial L_\Phi(x,u) = -\eta(x)\,\partial\Phi(-u^*) + (1-\eta(x))\,\partial\Phi(u^*).$$

Then u^* is a minimizer $\Leftrightarrow$ $\exists v_1 \in \partial\Phi(-u^*)$, $\exists v_2 \in \partial\Phi(u^*)$ s.t.

$$\eta(x)v_1 = (1-\eta(x))v_2.$$

If $u^*=0$, then since $\Phi'(0)$ exists and is greater than 0, we have $v_1 = v_2 = \Phi'(0) > 0$
$\Rightarrow \eta(x) = \frac{1}{2} \Rightarrow h^*(x) = 0$. This yields $h^*_\Phi(x) = 0 \Rightarrow h^*(x) = 0$.

To see the converse, suppose $h^*(x)=0$, i.e. $\eta(x)=\frac{1}{2}$. Then by definition of Φ being convex, $\frac{1}{2}\Phi(-u) + \frac{1}{2}\Phi(u) \geq \Phi(u+u) = \Phi(0)$ $\forall u \Rightarrow h^*_\Phi(x) = 0$ as well.

We may now assume WLOG that $\eta(x) \notin \{0, 1, \frac{1}{2}\}$. We first show that for any $u_1, u_2 \in \mathbb{R}$ with $u_1 < u_2$, and any two subgradients $v_1 \in \partial\Phi(u_1)$, $v_2 \in \partial\Phi(u_2)$, we have $v_1 \leq v_2$. Indeed, by definition of subgradient, we may bound

$$\left.\begin{aligned} \Phi(u_2) - \Phi(u_1) &\geq v_1(u_2-u_1) \\ \Phi(u_1) - \Phi(u_2) &\geq v_2(u_1-u_2) \end{aligned}\right\} \Rightarrow \begin{aligned} v_2(u_2-u_1) &\geq v_1(u_2-u_1) \\ \Rightarrow v_2 &\geq v_1. \end{aligned}$$

Now, if $u^* > 0$, then clearly $-u^* < u^*$. By the above argument, $v_1 \leq v_2$, where v_1, v_2 are the subgradient elements associated with $-u^*, u^*$, respectively.

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We cannot have $V_1 = V_2 \neq 0$, since this would imply by $\eta(x) V_1 = (1-\eta(x)) V_2$ that $\eta(x) = \frac{1}{2}$. Moreover, we cannot have $V_1 = V_2 = 0$, since ~~[illegible]~~ we know $V_2 \geq \Phi'(0) > 0$. So, we must have $V_2 > V_1$ with $V_2 > 0$.

This implies $\eta(x) > 1-\eta(x)$, so that $\eta(x) > \frac{1}{2}$ and thus $h^*(x) > 0$.

Conversely, suppose $h^*(x) > 0$; we want to show $u^* > 0$. Well, $h^*(x) > 0 \Rightarrow$ $\eta(x) > 1-\eta(x)$. As shown, we cannot have $V_1 = V_2 = 0$ or $V_1 = V_2 \neq 0$; since we also know $\eta(x) \neq 1$, it follows that $V_1 < V_2$. Thus, $-u^* \leq u^* \Rightarrow u^* \geq 0$.

Moreover, since $u^* = 0 \Rightarrow h^*(x) = 0$, we have $u^* > 0$. ∎

We may finally state and prove our main result.

<u>Theorem</u>: Let Φ be a convex, non-decreasing function. Suppose $\exists s \geq 1$ and $c > 0$ such that

$$\forall x \in X, \quad |h^*(x)|^s = \left|\eta(x) - \tfrac{1}{2}\right|^s \leq c^s \left[L_\Phi(x,0) - L_\Phi(x, h^*_\Phi(x))\right]$$

Then, for any hypothesis h,

$$R(h) - R^* \leq 2c \left[\mathcal{L}_\Phi(h) - \mathcal{L}^*_\Phi\right]^{\frac{1}{s}}$$

<u>Proof</u>: By convexity of Φ,

$$\Phi(-2h^*(x)h(x)) \leq \eta(x)\Phi(-h(x)) + (1-\eta(x))\Phi(h(x))$$
$$= L_\Phi(x, h(x)).$$

By our earlier Lemma, Jensen's inequality, and $h^*(x) = \eta(x) - \frac{1}{2}$, we have that

$$R(h) - R^* = \mathop{\mathbb{E}}_{x \sim D_X}\left(|2\eta(x) - 1| \mathbb{1}_{h(x)h^*(x) \leq 0} \right)$$

$$\leq \mathop{\mathbb{E}}_{x \sim D_X}\left(|2\eta(x) - 1|^s \mathbb{1}_{h(x)h^*(x) \leq 0} \right)^{\frac{1}{s}}$$

$$\leq 2c \mathop{\mathbb{E}}_{x \sim D_X}\left(\left(\Phi(0) - L_\Phi(x, h^*_\Phi(x)) \right) \mathbb{1}_{h(x)h^*(x) \leq 0} \right)^{\frac{1}{s}}$$

$$\leq 2c \mathop{\mathbb{E}}_{x \sim D_X}\left(\left(\Phi(-2h^*(x)h(x)) - L_\Phi(x, h^*_\Phi(x)) \right) \mathbb{1}_{h(x)h^*(x) \leq 0} \right)^{\frac{1}{s}}$$

$$\leq 2c \mathop{\mathbb{E}}_{x \sim D_X}\left(\left(L_\Phi(x, h(x)) - L_\Phi(x, h^*_\Phi(x)) \right) \mathbb{1}_{h(x)h^*(x) \leq 0} \right)^{\frac{1}{s}}$$

$$\leq 2c \mathop{\mathbb{E}}_{x \sim D_X}\left(L_\Phi(x, h(x)) - L_\Phi(x, h^*_\Phi(x)) \right)^{\frac{1}{s}}$$

$$= 2c \left[\mathcal{L}_\Phi(h) - \mathcal{L}^*_\Phi \right]^{\frac{1}{s}}. \blacksquare$$