Homework 1
MATH 123 - Spring 2023
Tufts University, Department of Mathematics
Due: January 26, 2023

1. Question 1

Recall that for a matrix $A = (A_{ij})_{i,j=1}^n \in \mathbb{R}^{n\times n}$ and a vector $x = (x_1, \dots, x_n)^T \in \mathbb{R}^{n\times 1}$, matrix-vector multiplication is defined as $(Ax)_i = \sum_{k=1}^n A_{ik}x_k$. Prove that matrix-vector multiplication is *linear*, i.e.

$$\forall x, y \in \mathbb{R}^{n\times 1}, A(x+y) = Ax + Ay$$

and

$$\forall \alpha \in \mathbb{R}, x \in \mathbb{R}^{n\times 1}, A(\alpha x) = \alpha Ax.$$

2. Question 2

Suppose $A \in \mathbb{R}^{n\times n}$ is symmetric and let $\mathbf{0} = (0, 0, \dots, 0) \in \mathbb{R}^n$ denote the vector of all zeros, understood as a column or row vector from context. Prove that if A has an eigenvalue of 0 with associated eigenvector $x \neq \mathbf{0}$, then A is not invertible. In particular, what can you say about the *null space of* A, namely $\{x \in \mathbb{R}^{n\times 1} \mid Ax = \mathbf{0}\}$?

3. Question 3

Define the ℓ^2 *norm of* $x \in \mathbb{R}^n$ to be

$$\|x\|_2 = \sqrt{\sum_{i=1}^n |x_i|^2}.$$

Prove that $\|\cdot\|_2$ is indeed a norm, that is, show that each of the following properties holds:

(1) $\forall x \in \mathbb{R}^n,\ \|x\|_2 \geq 0$.
(2) $\|x\|_2 = 0 \iff x = \mathbf{0}$.
(3) $\forall \alpha \in \mathbb{R}, x \in \mathbb{R}^n, \|\alpha x\|_2 = |\alpha|\|x\|_2$.
(4) $\forall x, y \in \mathbb{R}^n,\ \|x+y\|_2 \leq \|x\|_2 + \|y\|_2$.

Conditions (1) and (2) together refer to being *positive definite*. Condition (3) refers to *absolute homogeneity*. Condition (4) is often called the *triangle inequality*, owing to its geometric interpretation in the case $n = 2$.

4. Question 4

Suppose $(x_1^1, x_1^2), (x_2^1, x_2^2), \dots, (x_n^1, x_n^2) \subset \mathbb{R}^2$ are sampled from a line $x_2 = \alpha x_1$, for some $\alpha \in \mathbb{R}$. Prove that the (empirical) covariance matrix is rank at most 1, i.e. one of its eigenvalues is necessarily 0.

5. Question 5

In MATLAB, run 'HW1.m,' to get the data X_Circle, consisting of uniformly sampled points from the interior of a ball in $\mathbb{R}^2$, and X_Ellipse, consisting of uniformly sampled points near an ellipse in $\mathbb{R}^2$.

(a) In MATLAB, compute the two principal components of X_Circle from the definition in terms of the covariance matrix. Do not use the build-in 'pca' function. Do they have geometric significance for this data? Discuss.

(b) In MATLAB, compute the two principal components of X_Ellipse from the definition in terms of the covariance matrix. Do not use the build-in 'pca' function. Do they have geometric significance for this data? Discuss.