**Homework 9**
MATH 123 - Spring 2023
Tufts University, Department of Mathematics
Due: April 13, 2023

## Question 1

Download the data 'HW9_TwoClassSeparable.mat' to reveal data $\{x_i\}_{i=1}^n \subset \mathbb{R}^2$ together with labels $\{y_i\}_{i=1}^n$.

(a) Use the MATLAB 'fmincon.m' function to solve the dual problem to the hard margin SVM. Then, compute the relevant hyperplane parameters and display the optimal separating hyperplane. What is the margin of the optimal hyperplane?

(b) Let $F(w,b) = \|w\|_2^2 + \alpha \sum_{i=1}^n \max(0, 1 - y_i(w^T x_i + b))$ be soft-margin the hinge loss. Using MATLAB's 'fmincon.m' function, compute the solution to the dual problem for a range of $\alpha \geq 0$ values, and plot the corresponding hyperplanes. Are all the hyperplanes separating? Explain.

## Question 2

For labeled data $\{(x_i, y_i)\}_{i=1}^n$, let $F(w,b) = \|w\|_2^2 + \alpha \sum_{i=1}^n \max(0, 1 - y_i(w^T x_i + b))^2$ be the soft margin hinge loss.

(a) Prove that minimizing this loss is equivalent to minimizing $G(w,b,\xi) = \|w\|_2^2 + \alpha\|\xi\|_2^2$ subject to the hard constraints $y_i(w^T x_i + b) \geq 1 - \xi_i$. The variables $\xi = (\xi_1, \ldots, \xi_n)$ are called *slack variables.*

(b) Prove that the corresponding dual problem is maximized when $w = \frac{1}{2}\sum_{i=1}^n \lambda_i y_i x_i$ and $b = y_i(1 - \xi_i) - w^T x_i$, for any $\xi_i$ with $i > 0$.

## Question 3

Consider linearly separable data $\{(x_i, y_i)\}_{i=1}^n$, $x_i \in \mathbb{R}^d, y_i \in \{-1, 1\}$. Let $\{x \mid w^T x + b = 0\}$ be the margin-maximizing hyperplane that separates the data.

(a) Show that there are *at least* two support vectors, i.e. two points which minimize the distance between the maximum-margin hyperplane and the data.

(b) Could there be more than two support vectors? If so, show an example. If not, prove why not.

## Question 4

Use the dual optimization formulation to solve the following optimization problems, currently written in their primal form.

(a) Minimize $x^2 + y^2$ subject to the constraints $x \leq y, x \leq 1 - 2y$.

(b) Minimize $x^2 + y^2 + z^2$ subject to the constraints $x + y + z \geq 1$.