Lecture #4

· Note that the central limit theorem and laws of large numbers are statements about convergence of certain types of r.v.

Theorem (Weak Law of Large Numbers): Let $\{X_n\}_{n=1}^{\infty}$ be a sequence of iid r.v. with bounded expectation and variance. Let $\overline{X}_n = \frac{1}{n}\sum_{k=1}^{n} X_k$ be the empirical average of the first n observations. Then

$$\overline{X}_n \xrightarrow{\mathbb{P}} \mathbb{E}(X_1)$$

Proof: Let $\varepsilon > 0$. Then $\lim_{n\to\infty} \mathbb{P}\left(\left|\overline{X}_n - \mathbb{E}(X_1)\right| > \varepsilon\right)$

$$\overset{\text{Chebyshev}}{\leq} \lim_{n\to\infty} \frac{Var(\overline{X}_n)}{\varepsilon^2}$$

$$= \lim_{n\to\infty} \frac{\frac{1}{n}\cdot Var(X_1)}{\varepsilon^2}$$

$$= 0. \quad \blacksquare$$

Remark: If we assumed the $\{X_n\}_{n=1}^{\infty}$ were bounded, Hoeffding would give a bound like $\exp(-n\varepsilon^2)$ ... much faster!

Theorem (Central Limit Theorem): Let $\{X_n\}_{n=1}^{\infty}$ be a sequence of iid r.v. with bounded expectation and variance, call them $\mu = \mathbb{E}(X_1)$, $\sigma^2 = Var(X_1)$.

Then if $Z_n := \dfrac{\frac{1}{n}\sum_{k=1}^{n} X_k - \mu}{\sqrt{\sigma^2/n}} = \dfrac{\sqrt{n}\left[\bar{X}_n - \mu\right]}{\sigma}$, $\bar{X}_n = \frac{1}{n}\sum_{k=1}^{n} X_k$,

$Z_n \rightsquigarrow Z$, where $Z \sim N(0,1)$.

- Recalling that $Z_n \rightsquigarrow Z \Leftarrow F_{Z_n}(t) = F_Z(t)$ at all points (since $F_Z(t)$ is continuous), this gives us a convenient computational tool:

$$\lim_{n\to\infty} \mathbb{P}(Z_n \leq b) = \int_{-\infty}^{b} \frac{1}{\sqrt{2\pi}} \exp\left(\frac{-z^2}{2}\right) dz$$

$$\Rightarrow \lim_{n\to\infty} \mathbb{P}(Z_n \in [a,b]) = \int_{a}^{b} \frac{1}{\sqrt{2\pi}} \exp\left(\frac{-z^2}{2}\right) dz$$

$$\Rightarrow \text{For } n \text{ large}, \quad \mathbb{P}(Z_n \in [a,b]) \approx \int_{a}^{b} \frac{1}{\sqrt{2\pi}} \exp\left(\frac{-z^2}{2}\right) dz.$$

The quality of this approximation is important in practice, and can be controlled by the Berry-Essèen inequality

Theorem (Berry-Esséen): Let the notation be as in the CLT. Suppose $\mathbb{E}|X_1|^3 < \infty$. Then

$$\sup_z \left| \mathbb{P}(Z_{1n} \leq z) - \Phi(z) \right| \leq \frac{33}{4} \cdot \frac{\mathbb{E}|X_1 - \mu|^3}{\sqrt{n} \cdot \sigma^3}$$

$$= C \cdot \frac{1}{\sqrt{n}}, \quad \text{for } C \text{ a constant independent of } n.$$

- Proof of CLT & B.-E. are a little difficult; see Wasserman for CLT.
- Remarkably, the CLT can be used as the basis for analyzing a broad class of r.v.

Theorem: Suppose $g : \mathbb{R} \to \mathbb{R}$ is differentiable. Let $\bar{X}_n$ be s.t. $\frac{\sqrt{n}[\bar{X}_n - \mu]}{\sigma} \rightsquigarrow Z_1$. Then if $g'(\mu) \neq 0$,

$$\frac{\sqrt{n}\left[g(\bar{X}_n) - g(\mu)\right]}{|g'(\mu)|\sigma} \rightsquigarrow Z_1.$$

- This allows us to understand the asymptotic behavior of functions of empirical averages. It is called the Delta Method.

ex: Let $X_n \sim \text{Bern}(1/2)$. Then clearly the CLT applies to $\frac{1}{n}\sum_{k=1}^{n} X_k =: \bar{X}_n$, so that $\frac{\sqrt{n}[\bar{X}_n - \frac{1}{2}]}{1/4} \rightsquigarrow Z_1$. The Delta Method tells us

That $$\frac{\sqrt{n}\left[(\bar{X}_n)^2 - \frac{1}{4}\right]}{2 \cdot \frac{1}{2} \cdot \frac{1}{4}} \rightsquigarrow Z$$

$$\Rightarrow \frac{\sqrt{n}\left[(\bar{X}_n)^2 - \frac{1}{4}\right]}{1/4} \rightsquigarrow Z.$$

- It does not say "consider $\bar{X}_n^2$", but instead says "consider $(\bar{X}_n)^2$".